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Continuous Selections for the Metric Projection and Alternation

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In this paper we give a characterization of those n-dimensional subspaces of $C_0(X)$, where X are certain locally compact spaces, for which alternation-elements are unique. As a consequence we obtain a result on the existence of continuous, quasi-linear selections for the metric projection in $C_0(X)$, which represents a partial solution of a problem posed by Lazar *et al.* [*J. Functional Analysis 3* (1969), 193–216]. Furthermore, we establish a necessary condition for the existence of inner-radial-continuous selections for the metric projection in normed linear spaces. From this we deduce results on the nonexistence of inner-radial-continuous selections for the metric projection. Finally, we give a characterization of those exponential sums in $C[a, b]$ which admit an inner-radial-continuous selection for their metric projection.

INTRODUCTION

If G is a nonempty subset of a normed linear space E, then, for each x in E, the set $P_G(x) := \{g_0 \in G: \|x - g_0\| = \inf\{\|x - g\| : g \in G\}\}$ is called the set of *best approximations* of x from G. This defines a set-valued mapping P_G from E into 2^G which is called the *metric projection* onto G. A mapping s from E onto G is called a *continaous* (respectively, *inner-radial-continuous*) *selection* for P_G, if $s(x) \in P_G(x)$ for each $x \in E$, and $x_n \to x$ (respectively, $(x_n) \subset \{g_0 + a(x - g_0): 0 \leqslant a < 1\}$ with $x_n \to x$, where $g_0 \in P_G(x)$) imply $s(x_n) \to s(x)$. The concept of radial-continuity has been introduced by Brosowski and Deutsch [5, 6]. The set G is called *proximinal* (respectively, *Chebyshev*) if $P_G(x)$ contains at least one (respectively, exactly one) element for each x in E.

Continuity criteria for the set-valued metric projection and, in particular, selection properties have been investigated by many authors in recent years (see, e.g., Singer [22] and Vlasov [25]). In this paper we consider the question of existence of (inner-radial-) continuous selections for P_G.

Lazar *et al.* [13] gave the first characterization of those one-dimensional subspaces G of $C(X)$ which admit a continuous selection for P_G. They posed

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the problem to characterize the corresponding n-dimensional subspaces. This question has also been raised in the book of Holmes [10]. Results for $n > 1$ are known only in the case $X = [a, b]$. In Section 1 we give an existence theorem for continuous, quasi-linear selections for P_G, for a class of n-dimensional weak Chebyshev subspaces G in $C_0(X)$, where X is an arbitrary locally compact subset of the real line if $n \geqslant 2$, and show that the assumptions on G are essential in a certain sense. The key result (Theorem 1.2) in this section, which may be of independent interest, is a characterization of those n-dimensional subspaces in $C_0(X)$, where X is a locally compact subset of the real line if $n \geqslant 2$, for which each $f \in C_0(X)$ has a unique alternation element (Definition 1.1). In the particular case $X = [a, b]$, this has been proved by Nürnberger and Sommer [18]: their arguments, however, do not apply in the general situation. As a corollary we obtain the above-mentioned selection theorem (Corollary 1.3). In the particular case of continuous selections for P_G, Corollary 1.3 has been proved for X compact and $n = 1$ by Lazar *et al.* [13], and for $X = [a, b]$ and n arbitrary by Nürnberger and Sommer [18]. (A particular case is a result of Brown [7] for $X = [-1, 1]$ and $n = 5$).

Unlike existence of continuous selections for P_G, *each* n-dimensional subspace in any normed linear space admits an inner-radial-continuous selection for P_G, and more (see [17]). The situation, however, is completely different if we consider nonlinear sets G.

In Section 2 we give a necessary condition for the existence of inner-radial-continuous selections for P_G, where G is a proximinal subset in a normed linear space (Theorem 2.1). As a consequence, we get that, if G is the boundary of a ball in a normed linear space, then P_G has no inner-radial-continuous selection (Corollary 2.2). Finally we show that the exponential sums E_n in $C[a, b]$ allow an inner-radial-continuous selection for P_{E_n} if and only if $n = 1$ (Theorem 2.3).

Notation. For a normed linear space E and $x, y \in E$, $r > 0$, we denote $S(x, r) := \{y \in E : \|x - y\| = r\}$, $K(x, r) := \{y \in E : \|x - y\| < r\}$ and $[x, y] := \{ax + (1 - a)y : 0 \leqslant a \leqslant 1\}$. For $f \in C_0(X)$, $P \subset C_0(X)$ and $A \subset X$ we denote by $Z(f) := \{x \in X : f(x) = 0\}$, $Z(P) := \cap \{Z(p) : p \in P\}$, $f|_A$ the restriction of f to A and $bd\ A$ the boundary of A. If $g_1, \ldots, g_n$ are in a linear space then by span $\{g_1, \ldots, g_n\}$ we denote the linear hull of $\{g_1, \ldots, g_n\}$.

1. Linear Case

In this section we consider the question of the existence of continuous selections for P_G, where G is an n-dimensional subspace in a space of continuous functions.

For a locally compact Hausdorff space X let $C_0(X)$ be the space of all real-valued continuous functions f on X vanishing at infinity, i.e., for each $\epsilon > 0$ the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact, endowed with the norm $\|f\| = \sup\{|f(x)| : x \in X\}$ for each $f \in C_0(X)$. If X is compact then we denote $C_0(X)$ by $C(X)$.

In the following we consider n-dimensional subspaces G of $C_0(X)$, where X is a subset of the real line if $n \geq 2$. Furthermore the space X shall contain at least $n - 1$ points. The subspace G is called *weak Chebyshev*, if for each basis $\{g_1, ..., g_n\}$ of G there exists an $\epsilon = \pm 1$ such that for each n distinct points $x_1, ..., x_n$ in X ($x_1 < \cdots < x_n$, if $n \geq 2$) $\epsilon \det(g_i(x_j)) \geq 0$. The subspace G is called a *Chebyshev system* on Y, where Y is a subset of the real line, if for each basis $\{g_1, ..., g_n\}$ of G and each n distinct points $y_1, ..., y_n$ in Y $\det(g_i(y_j)) \neq 0$.

1.1. DEFINITION. If f is in $C_0(X)$ then g_f in $P_G(f)$ is called an *alternation-element* (A-element) of f, if there exist $n + 1$ distinct points $x_0, ..., x_n$ in X ($x_0 < \cdots < x_n$, if $n \geq 2$) such that $\epsilon(-1)^i (f - g_f)(x_i) = \|f - g_f\|$, $i = 0, 1, ..., n$, $\epsilon = \pm 1$. The points $x_0, ..., x_n$ are called *alternating extreme points* of $f - g_f$.

The next theorem, which may be of independent interest, is the key result in this section and represents a characterization of those n-dimensional subspaces in $C_0(X)$ for which we have uniqueness of alternation-elements.

1.2. THEOREM. *Let G be an n-dimensional subspace of $C_0(X)$, where X is a subset of the real line, if $n \geq 2$. Then the following statements are equivalent.*

(1) *G is weak Chebyshev and each $g \in G$, $g \neq 0$, has at most n distinct zeros.*

(2) *For each $f \in C_0(X)$ there exists exactly one alternation-element g_f in $P_G(f)$.*

Proof. We show that (1) implies (2). Therefore we assume that (1) holds. First we show that each $f \in C_0(X)$ has at least one A-element in $P_G(f)$.

Let $n = 1$. For f in G statement (2) is trivial. Therefore let f be in $C_0(X) \setminus G$ and $G = \mathrm{span}\{g_1\}$. Let y be the only zero of g_1. We choose a neighborhood basis (U_α) of y such that the sets U_α are open and small enough that g_1 is linearly independent on $K_\alpha = X \setminus U_\alpha$. The neighborhood basis (U_α) is a directed system, if we order it by inclusion. For each α we approximate f on K_α by $G_\alpha = \{g|_{K_\alpha} : g \in G\}$ with respect to the norm $\|h\|_\alpha = \sup\{|h(s)| : s \in K_\alpha\}$ for each $h \in C_0(K_\alpha)$. Since G is a Chebyshev system and weak Chebyshev on K_α, by Bram [4] for $P_{G_\alpha}(f) = \{g_\alpha\}$, there exist points x_0^α, x_1^α in K_α such that $\epsilon_\alpha(-1)^i (f - g_\alpha)(x_i^\alpha) = \|f - g_\alpha\|_\alpha$, $i = 0, 1$, $\epsilon_\alpha = \pm 1$. Since G is a finite-dimensional subspace by standard arguments $(\tilde{g}_\alpha)$ has a subnet converging to a function $g_f \in G$, where $g_\alpha = \tilde{g}_\alpha|_{K_\alpha}$ with $\tilde{g}_\alpha \in G$.

Passing to a subnet we also may assume that for each α we have $\epsilon_\alpha = \epsilon$ for some $\epsilon = \pm 1$. If X is compact then we may assume that for each $i = 0, 1$ (x_i^α) has a subnet converging to a point $x_i \in X$. If not, since X is locally compact, X can be imbedded in its one point compactification $X \cup \{\infty\}$ and $C_0(X)$ may be considered as a subspace of $C(X \cup \{\infty\})$ by defining $h(\infty) = 0$ for each $h \in C_0(X)$. Therefore we may assume that for $i = 0, 1$ (x_i^α) has a subnet converging to a point $x_i \in X \cup \{\infty\}$. Passing to subnets and taking limits we get $\epsilon(-1)^i (f - g_f)(x_i) = \|f - g_f\|$, $i = 0, 1$, $\epsilon = \pm 1$. The points x_i, $i = 0, 1$, cannot be equal to ∞, since $\|f - g_f\| > 0$. Furthermore, since for each $g \in G$ we have $\|f - g\|_{K_\alpha} \leqslant \|f - g_\alpha\|_\alpha$, we get by taking limits $\|f - g\| \leqslant \|f - g_f\|$ for each $g \in G$, i.e., $g_f \in P_G(f)$. Therefore g_f is an A-element of f.

If $n \geqslant 2$, then since G is weak Chebyshev by a result of Deutsch *et al.* [8], it follows that for each $f \in C_0(X)$ there exists an A-element of f.

Now we show that for each $f \in C_0(X)$ there exists exactly one A-element in $P_G(f)$. This is done by contradiction. Assume that there exists a function $f \in C_0(X) \setminus G$ which has two distinct A-elements g_0, g_1 in $P_G(f)$. We may assume that $g_1 = 0$. Therefore there exist $n - 1$ distinct points $x_0, \ldots, x_n$ (respectively, $y_0, \ldots, y_n$) in X ($x_0 < \cdots < x_n$ (respectively, $y_0 < \cdots < y_n$), if $n \geqslant 2$) such that

(a) $(-1)^i f(x_i) = \|f\|$, $i = 0, \ldots, n$ (*respectively*, $\epsilon(-1)^i (f - g_0)(y_i) = \|f - g_0\|$, $i = 0, \ldots, n$, $\epsilon = \pm 1$). From this it follows

(b) $(-1)^i g_0(x_i) \geqslant 0$ and $\epsilon(-1)^i g_0(y_i) \leqslant 0$, $i = 0, \ldots, n$.

First we consider the case $n = 1$. We may assume that $g_0(x_0) = 0$ or $g_0(x_1) = 0$ (respectively, $g_0(y_0) = 0$ or $g_0(y_1) = 0$), otherwise we would have a contradiction to the fact that G is weak Chebyshev. Let $\epsilon = 1$. We first consider the case when $g_0(x_0) = 0 = g_0(y_0)$. If $g_0(x_1) = 0$ or $g_0(y_1) = 0$, then g_0 has two distinct zeros, and if $g_0(x_1) < 0$ and $g_0(y_1) > 0$, we have a contradiction to the fact that G is weak Chebyshev. Now we consider the case when $g_0(x_0) = 0 = g_0(y_1)$. Then $x_0 \neq y_1$, otherwise by (a) $g_0(x_0) = g_0(y_1) > 0$, but then g_0 has two distinct zeros, which is not possible. The other cases follow analogously. Similar arguments hold in the case $\epsilon = -1$.

Now we consider the case $n \geqslant 2$. First we show that

(c) there does not exist a function $g \in G$, $g \neq 0$, with the property that there exist $n - 3$ distinct points $t_1 < \cdots < t_{n+3}$ such that

$$(-1)^{i+1} g(t_i) \geqslant 0, \; i = 1, \ldots, n - 3.$$

Assume that there exists a function $g \in G$, $g \neq 0$, as in (c). Since each $g \in G$, $g \neq 0$, has at most n distinct zeros, there exists a point $y_1 \in \{t_1, \ldots, t_{n+1}\}$ such that G is a Chebyshev system on $\{t_1, \ldots, t_{n+1}\} \setminus \{y_1\}$. Set $\{s_1, \ldots, s_n\} = \{t_1, \ldots, t_{n+1}\} \setminus \{y_1\}$, such that $s_1 < \cdots < s_n$, and $y_2 = t_{n+2}$, $y_3 = t_{n+3}$. Since G is a

Chebyshev system on $\{s_1, \ldots, s_n\}$ there exists a basis $\{g_1, \ldots, g_n\}$ of G such that for each $i \in \{1, \ldots, n\}$ we have $g_i(s_j) = 0$, if $j \neq i$, and $g_i(s_i) = 1$, if $s_i = t_j$ with j odd (respectively, $g_i(s_i) = -1$, if $s_i = t_j$ with j even).

Then $g = a_1 g_1 + \cdots + a_n g_n$ with $a_1, \ldots, a_n \geqslant 0$ and the scalars a_i are not all zero. We define

$$D = \begin{vmatrix} g_1(s_1) & \cdots & g_1(s_n) \\ \vdots & & \vdots \\ g_n(s_1) & \cdots & g_n(s_n) \end{vmatrix},$$

and for each $i \in \{1, \ldots, n\}$,

$$D_i = \begin{vmatrix} g_1(s_1) \cdots g_1(s_j)\, g_1(y_1)\, g_1(s_{j+1}) \cdots g_1(s_{i-1})\, g_1(s_{i+1}) \cdots g_1(s_n) \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ g_n(s_1) \cdots g_n(s_j)\, g_n(y_1)\, g_n(s_{j+1}) \cdots g_n(s_{i-1})\, g_n(s_{i+1}) \cdots g_n(s_n) \end{vmatrix},$$

where $s_1 < \cdots < s_j < y_1 < s_{j+1} < \cdots < s_{i-1} < s_{i+1} < \cdots < s_n$. Since G is weak Chebyshev, we have $DD_i \geqslant 0$, $i = 1, \ldots, n$, and it is easy to verify, that from this it follows that for each $i \in \{1, \ldots, n\}$ $g_i(y_1) \leqslant 0$, if $y_1 = t_j$ with j odd (respectively, $g_i(y_1) \geqslant 0$, if $y_1 = t_j$ with j even).

Therefore, since $g(t_j) \geqslant 0$, if j even (respectively, $g(t_j) \leqslant 0$, if j is odd) we get $g(y_1) = a_1 g_1(y_1) + \cdots + a_n g_n(y_1) = 0$. Since the real numbers $a_1 g_1(y_1), \ldots, a_n g_n(y_1)$ have the same sign, it follows that

(i) for each $i \in \{1, \ldots, n\}$ with $a_i \neq 0$ we have $g_i(y_1) = 0$. Now we define for each $i \in \{1, \ldots, n\}$ and each $t \geqslant s_n$, where $t \in T$,

$$D_i(t) = \begin{vmatrix} g_1(s_1) \cdots g_1(s_{i-1})\, g_1(s_{i+1}) \cdots g_1(s_n)\, g_1(t) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ g_n(s_1) \cdots g_n(s_{i-1})\, g_n(s_{i+1}) \cdots g_n(s_n)\, g_n(t) \end{vmatrix}.$$

Since G is weak Chebyshev, we have for each $i \in \{1, \ldots, n\}$ and each $t \geqslant s_n$, where $t \in T$, $DD_i(t) \geqslant 0$, and it is easy to verify that from this it follows that

(ii) if $y_2 = t_j$ with j odd (respectively, j even), then for each $t \geqslant s_n$, where $t \in T$, $g_i(t) \geqslant 0$ (respectively, $g_i(t) \leqslant 0$) for $i \in \{1, \ldots, n\}$ with $s_i < y_1$ and $g_i(t) \leqslant 0$ (respectively, $g_i(t) \geqslant 0$) for each $i \in \{1, \ldots, n\}$ with $s_i > y_1$. Now we define for each $i, j \in \{1, \ldots, n\}$ with $i < j$ the determinant D_{ij} by

$$\begin{vmatrix} g_1(s_1) \cdots g_1(s_{i-1})\, g_1(s_{i+1}) \cdots g_1(s_{j-1})\, g_1(s_{j+1}) \cdots g_1(s_n)\, g_1(y_2)\, g_1(y_3) \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ g_n(s_1) \cdots g_n(s_{i-1})\, g_n(s_{i+1}) \cdots g_n(s_{j-1})\, g_n(s_{j+1}) \cdots g_n(s_n)\, g_n(y_2)\, g_n(y_3) \end{vmatrix}.$$

Using (ii), a simple calculation shows that for each $i, j \in \{1, \ldots, n\}$ with $i < j$ $DD_{ij} = |g_i(y_2)|\, |g_j(y_3)| - |g_j(y_2)|\, |g_i(y_3)|$. Since G is weak Chebyshev it follows that

(iii) for each $i, j \in \{1,..., n\}$ with $i < j$

$$|g_j(y_2)| \, |g_i(y_3)| \geqslant |g_i(y_2)| \, |g_j(y_3)| .$$

We assume that $y_2 = t_j$ with j odd (The other case follows analogously). By scaling with positive scalars we may assume that for each $a_i \neq 0$ $|g_i(y_3)| = 1$, since if $g_i(y_3) = 0$ for some $i \in \{1,..., n\}$ with $a_i \neq 0$ the function g_i would have $n + 1$ distinct zeros at $(\{s_1 ,..., s_n\} \cup \{y_1 , y_3\})\backslash\{s_i\}$, which is a contradiction. We remark that after this procedure statements (i)–(iii) remain valid. Then from (iii) it follows that

(iv) for each $i, j \in \{1,..., n\}$ with $i < j$, $|g_i(y_2)| \leqslant |g_j(y_2)|$. Set $I_1 = \{i: s_i < y_1 , a_i \neq 0\}$, $I_2 = \{i : s_i > y_1 , a_i \neq 0\}$ and let k be such that $|g_k(y_2)| = \min\{|g_j(y_2)| : j \in I_2\}$. Then from (iv) it follows that

$$0 \leqslant g(y_2) = \sum_{i\in I_1} a_i \, |g_i(y_2)| - \sum_{j\in I_2} a_j \, |g_j(y_2)|$$

$$\leqslant \Big(\sum_{i\in I_1} a_i - \sum_{j\in I_2} a_j\Big) \, |g_k(y_2)| = g(y_3) \, |g_k(y_2)| \leqslant 0.$$

If $I_1 = \varnothing$ or $I_2 = \varnothing$ then $g_k(y_2) = 0$ and the function g_k has $n + 1$ distinct zeros at $(\{s_1 ,..., s_n\} \cup \{y_1 , y_2\})\backslash\{s_k\}$, which is a contradiction. Therefore we may assume that $I_1 \neq \varnothing$ and $I_2 \neq \varnothing$. Now, if there exists a number $i \in I_1$ with $|g_i(y_2)| < |g_k(y_2)|$; then we have

$$0 \leqslant g(y_2) = \sum_{i\in I_1} a_i \, |g_i(y_2)| - \sum_{j\in I_2} a_j \, |g_j(y_2)|$$

$$< \Big(\sum_{i\in I_1} a_i - \sum_{j\in I_2} a_j\Big) \, |g_k(y_2)| \leqslant 0,$$

which is a contradiction.

Otherwise for each $i \in I_1$ we have $|g_i(y_2)| = |g_k(y_2)|$ and therefore from (ii) it follows that $g_i(y_2) = -g_k(y_2)$ and $g_i(y_3) = 1 = -g_k(y_3)$. But then from (i) it follows that the function $g_i + g_k$ is not identically zero and has at least $n + 1$ distinct zeros at $(\{s_1 ,..., s_n\} \cup \{y_1 , y_2 , y_3\})\backslash\{s_i , s_k\}$, which again is a contradiction. This shows (c).

Now let $\epsilon = 1$. If $x_i = y_i$, $i = 0,..., n$, then by (b) g_0 has $n + 1$ distinct zeros at $x_0 ,..., x_n$, which is a contradiction. Therefore we may assume that $x_i < y_i$ for some i. If $y_j < x_j$ for some j then by (b) the function g_0 has alternating sign at the $n + 3$ points $y_0 < \cdots < y_j < x_j < \cdots < x_i < y_i < \cdots < y_n$ if $j < i$ (respectively, at $x_0 < \cdots < x_i < y_i < \cdots < y_j < x_j < \cdots < x_n$ if $j > i$). But this is a contradiction to (c). If $x_{i+2} < y_i$ for some i then by (b) again g_0 has alternating sign at the $n + 4$ points $x_0 < \cdots < x_{i+2} < y_i < \cdots < y_n$, which is a contradiction to (c). Therefore we have $x_i \leqslant y_i \leqslant$

x_{i+2}, $i = 0,\ldots, n$, where the points x_{n+1} and x_{n+2} are omitted. Now we order the points $x_0,\ldots, x_n$, $y_0,\ldots, y_n$ and get points $s_1 < s_2 < \tilde{s}_2 < \cdots < s_{n-1} < \tilde{s}_{n+1} < s_{n-2}$ such that

$$(-1)^{i-1} g_0(s_i) \geq 0, \; i = 1,\ldots, n-2 \text{ and } (-1)^{i-1} g_0(\tilde{s}_i) \geq 0, \; i = 2,\ldots, n+1.$$

Then by (a) and (b) we have the following.

(d) If $s_i = \tilde{s}_i$ for some $i \in \{2,\ldots, n\}$ then $g_0(s_i) = g_0(\tilde{s}_i) = 0$ and $\tilde{s}_{i-1} < s_i = \tilde{s}_i < s_{i+1}$. If $\tilde{s}_i = s_{i+1}$ for some $i \in \{2,\ldots, n\}$ then $g_0(\tilde{s}_i) = g_0(s_{i+1}) = 0$ and $s_i < \tilde{s}_i = s_{i+1} < \tilde{s}_{i+1}$. If $s_1 = s_2$ then $g_0(s_1) = 0 = g_0(s_2)$ and $s_2 < \tilde{s}_2$. If $\tilde{s}_{n+1} = s_{n+2}$ then $g_0(\tilde{s}_{n+1}) = 0 = g_0(s_{n-2})$ and $s_{n+1} < \tilde{s}_{n+1}$.

In the following argumentation, where we show that our assumption leads to contradictions, (d) will be essential.

If $s_1 = s_2$, $\tilde{s}_2 = s_3,\ldots, \tilde{s}_{n+1} = s_{n+2}$, then g_0 has $n+1$ distinct zeros at $\{s_1, s_3,\ldots, s_{n+2}\}$, which is a contradiction. Otherwise we consider the first $n+1$ points $u_1,\ldots, u_{n+1}$ from $\{s_1,\ldots, s_{n+2}, \tilde{s}_2,\ldots, \tilde{s}_{n+1}\}$ for which we have $u_1 < \cdots < u_{n+1}$ and $(-1)^{i-1} g_0(u_i) \geq 0$, $i = 1,\ldots, n+1$. Since each $g \in G$, has at most n distinct zeros, there exists a point $y \in \{u_1,\ldots, u_{n+1}\}$ such that G is a Chebyshev system on $\{u_1,\ldots, u_{n+1}\}\setminus\{y\}$. Analogously as in the proof of (c)(i) we can show that there exists a function $g_k \in G$, $g_k \neq 0$, such that $g_k(y) = 0$ and therefore g_k has n distinct zeros at $\{u_1,\ldots, u_{n+1}\}\setminus\{u_k\}$.

If $y = u_1$, then there exist again $n+1$ distinct points $v_1,\ldots, v_{n+1}$ such that $\{v_1,\ldots, v_{n+1}\} = \{u_2,\ldots, u_{n+1}\} \cup \{s_{n+2}\}$, $v_1 < \cdots < v_{n+1}$ and $(-1)^i g_0(v_i) \geq 0$, $i = 1,\ldots, n+1$. Again concluding as in the proof of (ci) we can show that g_k has a further zero in s_{n+2} and therefore at least $n+1$ distinct zeros, which is a contradiction.

If $y \neq u_1$ then we conclude as follows: In this case $y = \tilde{s}_i$ or $y = s_i$ for some $i \in \{2,\ldots, n+1\}$. If $s_i \neq \tilde{s}_i$, then $g_0(y) \neq 0$, but analogously as in the proof of (ci) we can show that $g_0(y) = 0$, which leads to a contradiction.

If $s_i = \tilde{s}_i$, then there exists again a set of $n+1$ distinct points $\{v_1,\ldots, v_{n+1}\}$, containing $\{u_1,\ldots, u_{n+1}\}\setminus\{y\}$, but not containing the point y, such that $v_1 < \cdots < v_{n+1}$ and $\sigma(-1)^{i-1} g_0(v_i) \geq 0$, $i = 1,\ldots, n+1$, $\sigma = \pm 1$. Again concluding as in the proof of (ci) we can show that g_k has a further zero in $\{v_1,\ldots, v_{n+1}\}\setminus(\{u_1,\ldots, u_{n+1}\} \cup \{y\})$ and therefore at least $n+1$ distinct zeros, which is a contradiction.

Now let $\epsilon = -1$. If $y_i < x_{i+1}$ for some i, then by (b), g_0 has alternating sign at the $n+3$ points $y_0 < \cdots < y_i < x_{i+1} < \cdots < x_n$, which is a contradiction to (c). If $x_{i-1} < y_i$ for some i, then by (b) g_0 has alternating signs at the $n+3$ points $x_0 < \cdots < x_{i-1} < y_i < \cdots < y_n$, which again is a contradiction to (c). Therefore we have $x_{i-1} \leq y_i \leq x_{i+1}$, $i = 0,\ldots, n$, where the points x_{-1} and x_{n+1} are omitted. We order the points $x_0,\ldots, x_n$, $y_0,\ldots, y_n$ and get points $s_1 < \tilde{s}_1 < \cdots < s_{n+1} < \tilde{s}_{n+1}$ such that $(-1)^{i-1} g_0(s_i) \geq 0$, $i = 1,\ldots,$

$n-1$, and $(-1)^{i-1} g_0(\tilde{s}_i) \geqslant 0$, $i = 1,..., n-1$. Then by (a) and (b) we have the following.

(e) If $s_i = \tilde{s}_i$ for some $i \in \{1,..., n\}$, then $g_0(s_i) = g_0(\tilde{s}_i) = 0$ and $\tilde{s}_{i-1} < s_i = \tilde{s}_i < s_{i+1}$. If $\tilde{s}_i = s_{i+1}$ for some $i \in \{1,..., n\}$, then $g_0(\tilde{s}_i) = 0 = g_0(s_{i+1})$ and $s_i < \tilde{s}_i = s_{i+1} < \tilde{s}_{i+1}$.

Statement (e) will be essential in the following argumentation. We consider the $n+1$ distinct points $s_1,..., s_{n+1}$ for which $(-1)^{i-1} g_0(s_i) \geqslant 0$, $i = 1,..., n+1$. Since each $g \in G$, $g \neq 0$, has at most n distinct zeros, there exists a point $y \in \{s_1,..., s_{n+1}\}$, say $y = s_i$, such that G is a Chebyshev system on $\{s_1,..., s_{n+1}\} \setminus \{y\}$. If $s_i = \tilde{s}_i$, then $g_0(y) \neq 0$, but analogously as in the proof of (ci) we can show that $g_0(y) = 0$, which leads to a contradiction. If $s_i \neq \tilde{s}_i$, then analogously as in the proof of (ci) we can show that there exists a function $g_k \in G$, $g_k \neq 0$, with n distinct zeros at $\{s_1,..., s_{n+1}\} \setminus \{s_k\}$. Furthermore there exists a set of $n+1$ distinct points $\{v_1,..., v_{n+1}\}$, containing $\{s_1,..., s_{n+1}\} \setminus \{y\}$, but not containing the point y, such that $v_1 < \cdots < v_{n+1}$ and $\sigma(-1)^{i-1} g_0(v_i) \geqslant 0$, $i = 1,..., n+1$, $\sigma = \pm 1$. Then again concluding as in the proof of (ci) we can show that g_k has a further zero in $\{v_1,..., v_{n+1}\} \setminus (\{s_1,..., s_{n+1}\} \setminus \{y\})$ and therefore g_k has at least $n+1$ distinct zeros, which is a contradiction. This shows that (1) implies (2).

Now we show that (2) implies (1). First we show that (2) implies that G is weak Chebyshev. Let $n = 1$. If X is compact then we set $f = 1$. Since by (2) there exists an alternation-element $g_1 \in P_G(f)$, the function g_1 can not be identically zero and $g_1 \geqslant 0$. Therefore $G = \text{span}\{g_1\}$ is weak Chebyshev. Therefore assume that G is not compact. Since X is locally compact it can be imbedded in its one point compactification $X \cup \{\infty\}$ and $C_0(X)$ may be considered as a subspace of $C(X \cup \{\infty\})$ by defining $h(\infty) = 0$ for each $h \in C_0(X)$. Now we choose a neighborhood basis (U_α) of ∞ such that the U_α's are open. The neighborhood basis (U_α) is a directed system if we order it by inclusion. By Tietze's Lemma for each α there exists a function $f_\alpha \in C(X \cup \{\infty\})$ such that $f_\alpha = 1$ on $X \setminus U_\alpha$, $f_\alpha(\infty) = 0$ and $0 \leqslant f_\alpha \leqslant 1$. Since for each α we have $f_\alpha|_X \in C_0(X)$ and from (2) it follows that there exists an alternation-element $g_\alpha \in P_G(f_\alpha|_X)$, obviously $g_\alpha \geqslant 0$ on $X \setminus U_\alpha$ and $g_\alpha \neq 0$, otherwise g_α would not be an alternation-element of f_α. Therefore by scaling we may assume that for each α $\|g_\alpha\| = 1$. Since G is finite dimensional by standard arguments (g_α) has a subnet converging to a function $g_1 \in G$, $g_1 \neq 0$, such that $g_1 \geqslant 0$. This shows that $G = \text{span}\{g_1\}$ is weak Chebyshev. If $n \geqslant 2$ then by a result of Deutsch *et al.* [8] it follows that if for each $f \in C_0(X)$ there exists an alternation-element $g_f \in P_G(f)$, then G is weak Chebyshev. This shows that (2) implies that G is weak Chebyshev.

Now we show that (2) implies that each $g \in G$, $g \neq 0$, has at most n distinct zeros in X. Assume that there exists a function $g_0 \in G$, $g_0 \neq 0$, which has $n+1$ distinct zeros $x_0,..., x_n$ in $X(x_0 < \cdots < x_n$, if $n \geqslant 2)$. By scaling we may assume that $\|g_0\| = 1$. We show that there exists a function f in $C_0(X)$, which has 0 and g_0 as A-elements. Since X is a Hausdorff space there exist neighborhoods U_i of x_i, $i = 0, \ldots, n$, which are disjoint. Then there exists a function f in $C_0(X)$ with the properties $\|f\| = 1$, $(-1)^i f(x_i) = 1$, $i = 0, \ldots, n$, $0 \leq f(x) \leq \min\{1 + g_0(x), 1\}$ for $x \in U_i$, if $f(x_i) = 1$, $\max\{-1 + g_0(x), -1\} \leq f(x) \leq 0$ for $x \in U_i$, if $f(x_i) = -1$ and $f(x) = 0$ for $x \in X \setminus \bigcup\{U_i : i = 0, \ldots, n\}$. Then the functions f and $f - g_0$ obviously have $n + 1$ alternating extreme points $x_0, \ldots, x_n$ and $\|f\| = 1 = \|f - g_0\|$. Furthermore we have that 0 and g_0 are in $P_G(f)$, otherwise there exists a function $g \in G$ such that $\|f - g\| < \|f\|$. This implies $(-1)^i (f - g)(x_i) < (-1)^i f(x_i)$ and therefore $(-1)^i g(x_i) > 0$, $i = 0, \ldots, n$. For $n = 1$ this obviously is a contradiction and for $n \geq 2$ we also get a contradiction, since by a result of Deutsch *et al.* [8] and Zielke [26], in a weak Chebyshev subspace G there does not exist a function $\tilde{g} \in G$ and distinct points $x_0 < \cdots < x_n$ in X such that $(-1)^i \tilde{g}(x_i) > 0$, $i = 0, \ldots, n$. Therefore we have shown that (2) implies (1), and this completes the proof.

In the special case $X = [a, b]$, Theorem 1.2 has been proved in Nürnberger and Sommer [18]. Their methods, however, do not apply to the general situation of Theorem 1.2.

Let E be a real vector space, G a subspace of E and s a mapping from E onto G. Then s is called *quasi-linear*, if for each $f \in E$, $g \in G$ and real numbers a and b we have $s(af + bg) = as(f) + bg$.

Using Theorem 1.2 we now are in position to prove the following result on the existence of continuous, quasi-linear selections for P_G.

1.3. COROLLARY. *Let G be an n-dimensional weak Chebyshev subspace of $C_0(X)$, where X is a subset of the real line if $n \geq 2$, such that each $g \in G$, $g \neq 0$, has at most n distinct zeros in X. Then there exists a continuous, quasi-linear selection for P_G.*

Proof. From the properties of G and Theorem 1.2 it follows that each f in $C_0(X)$ has a unique A-element g_f in $P_G(f)$. We define the selection s by $s(f) = g_f$ for each f in $C_0(X)$.

(1) We show that s is continuous. If not, since G is finite dimensional, there exist $f_m \to f$ and $s(f_m) \to g$ with $s(f) \neq g$ and $g \in P_G(f)$. Furthermore for each m there exist $n + 1$ distinct points $x_0^m, \ldots, x_n^m$ in X ($x_0^m < \cdots < x_n^m$ if $n \geq 2$) such that $\epsilon_m (-1)^i (f_m - s(f_m))(x_i^m) = \|f_m - s(f_m)\|$, $i = 0, \ldots, n$, $\epsilon_m = \pm 1$. Similarly, as in the proof of Theorem 1.2, by passing to subsequences and taking limits we get $\epsilon(-1)^i (f - g)(x_i) = \|f - g\|$, $i = 0, \ldots, n$, $\epsilon = \pm 1$, where $x_0, \ldots, x_n$ are distinct points in X ($x_0 < \cdots < x_n$ if $n \geq 2$). Furthermore, since for each m $s(f_m) \in P_G(f_m)$ and $f_m \to f$ we have $g \in P_G(f)$, as it is well known. Therefore g is an A-element of f with $s(f) \neq g$, which is a contradiction to the uniqueness of A-elements.

(2) We show that s is quasi-linear. Let $f \in C_0(X), g \in G$ and real numbers a and b be given. Since by definition $s(f)$ is an A-element of f, there exist distinct points $x_0, \ldots, x_n$ in X ($x_0 < \cdots < x_n$ if $n \geqslant 2$) such that $\epsilon(-1)^i (f - s(f))(x_i) = \|f - s(f)\|$, $i = 0, \ldots, n$, $\epsilon = \pm 1$. Then

$$\begin{aligned}
&\epsilon(-1)^i (af + bg - (as(f) + bg))(x_i) \\
&\quad = a\epsilon(-1)^i (f - s(f))(x_i) = a \|f - s(f)\| = \tilde{\epsilon} \|af - as(f)\| \\
&\quad = \tilde{\epsilon} \|af + bg - (as(f) + bg)\|, \qquad i = 0, \ldots, n, \tilde{\epsilon} = \pm 1, \epsilon = \pm 1.
\end{aligned}$$

Furthermore, as it is well known, we have $as(f) + bg \in aP_G(f) + bg = P_G(af + bg)$. Therefore $as(f) + bg$ is an A-element of $af + bg$. But since by definition $s(af + bg)$ is also an A-element of $af + bg$, if follows from the uniqueness of A-elements that $s(af + bg) = as(f) + bg$. This completes the proof.

We remark that for $n = 1$ ($G = \text{span}\{g_1\}$) Corollary 1.3 also holds, if we only assume that G is weak Chebyshev and $bd\, Z(g_1)$ contains at most one point. Because in this case we consider the metric projection from $C_0(\tilde{X})$ onto the restriction of G to $\tilde{X}$, where $\tilde{X} := (X \setminus Z(g_1)) \cup bd\, Z(g_1)$, and extend the existing continuous selection (according to Corollary 1.3) by zero to X.

Corollary 1.3 has been proved for continuous selections of P_G by Lazar *et al.* [13] for X compact and $n = 1$, and by Nürnberger and Sommer [18] for $X = [a, b]$ and n arbitrary, from which a result of Brown [7] for $X = [-1, 1]$ and $n = 5$ follows, using different kinds of approaches. Nevertheless their arguments do not apply to the general situation of Corollary 1.3.

In the case $X = [a, b]$ Corollary 1.3 was the crucial key in Nürnberger and Sommer [19] to give a complete characterization of continuous selections of the metric projection for spline functions.

We remark that the conditions on G in Corollary 1.3 are essential in a certain sense, because in Nürnberger [16], it is shown that a necessary condition for an n-dimensional subspace G in $C[a, b]$, which admits a continuous selection for P_G, is that G is weak Chebyshev. Furthermore Sommer [23] has shown that a necessary condition for an n-dimensional weak Chebyshev subspace G in $C[a, b]$, for which no $g \in G$, $g \neq 0$, vanishes on an interval and which admits a continuous selection for P_G, is that each $g \in G$, $g \neq 0$, has at most n distinct zeros in $[a, b]$.

Finally we give some examples of n-dimensional subspaces G in $C_0(X)$, which fulfill the condition in Theorem 1.2 and Corollary 1.3.

1.7. EXAMPLES. (1) Several examples of n-dimensional subspaces in $C[a, b]$, which fulfill the conditions in Theorem 1.2 and Corollary 1.3, can be found in Brown [7] and Nürnberger and Sommer [18]. A standard example is

$$G = \text{span}\{g_1, \ldots, g_n\} \subset C[0, 1], \qquad \text{where } g_i(x) = x^i, i = 1, \ldots, n.$$

(2) Let $\{g_1,\dots,g_n\}$ be a Chebyshev system of continuous real-valued functions on $\mathbb{R}$ and let g_0 be in $C_0(\mathbb{R})$ such that $g_0g_i\in C_0(\mathbb{R})$, $i=1,\dots,n$, and $g_0(y)=0$ for some $y\in\mathbb{R}$ and $g_0(x)>0$ for $x\in\mathbb{R}\setminus\{y\}$ (e.g., $g_i(x)=x^{i-1}$, $i=1,\dots,n$, and $g_0(x)=(1/e)\,x^2$ for $x\in[-1,1]$ and $g_0(x)=1/e^{x^2}$ elsewhere). Then $G=\operatorname{span}\{g_0g_1,\dots,g_0g_n\}$ is an n-dimensional subspace of $C_0(\mathbb{R})$, and by standard arguments (compare Jones and Karlovitz [11]) G fulfills the conditions in Theorem 1.2 and Corollary 1.3, and therefore we have the uniqueness of alternation-elements and the existence of a continuous, quasi-linear selection for P_G. The same holds, if we consider the restriction of G to any closed subset of the real line, containing at least $n+1$ distinct points. Similar arguments give us examples of n-dimensional subspaces of $C_0(X)$ for arbitrary (not necessarily closed) subsets of the real line.

2. Nonlinear Case

As we have seen in Section 1, not every n-dimensional subspace G in a normed linear space admits a continuous selection for P_G. However, this (and even more) is true for inner-radial-continuous selections for P_G (see [17]). But the situation is completely different, if we consider nonlinear sets, as we will see in the following.

First we give a necessary condition for the existence of inner-radial-continuous selections for P_G in arbitrary normed linear spaces.

A set S in a normed linear space E is called *star shaped* about x_0 in E, if for each x in S we have $[x_0, x]\subset S$.

2.1. Theorem. *Let G be a proximinal subset in a normed linear space E. If there exists an inner-radial-continuous selection s for P_G then for each x in E and each g_0 in $P_G(x)$ we have*

$$[g_0, s(x)]\subset S(x, d(x, G)).$$

Proof. Let s be an inner-radial-continuous selection for P_G and $x\in E$, $g_0\in P_G(x)$, $0\le a\le 1$. We show that for each $0\le b<1$ we have

(1) $s(g_0+b(x-g_0))\in S(x, d(x,G))\cap S(g_0+b(x-g_0), d(g_0+b(x-g_0), G))$. Let $0\le b<1$ be given. Then, of course, $s(g_0+b(x-g_0))$ is in $S(g_0+b(x-g_0), d(g_0+b(x-g_0), G))$. Therefore it remains to show that $s(g_0+b(x-g_0))$ is in $S(x, d(x,G))$. Since obviously $d(x,G)\le\|x-s(g_0+b(x-g_0))\|$ we show that $\|x-s(g_0+b(x-g_0))\|\le d(x,G)$. Assume that

(2) $\|x-s(g_0+b(x-g_0))\|>d(x,G)$. Since $g_0\in P_G(x)$, by the proof of Lemma 2.1 in Singer [21, pp. 364], $g_0\in P_G(g_0+b(x-g_0))$. Furthermore

(3) $bd(x, G) = b\|x - g_0\| = \|g_0 + b(x - g_0) - g_0\| = d(g_0 + b(x - g_0), G)$.

Then by (2) and (3) it follows that

$$\begin{aligned}
&\|g_0 + b(x - g_0) - s(g_0 + b(x - g_0))\| \\
&\quad = \|(x - g_0 - b(x - g_0)) - (x - s(g_0 + b(x - g_0))\| \\
&\quad \geqslant \|x - s(g_0 + b(x - g_0))\| - \|x - g_0 - b(x - g_0)\| \\
&\quad > d(x, G) - (1 - b)\, d(x, G) = bd(x, G) = d(g_0 + b(x - g_0), G).
\end{aligned}$$

But this is a contradiction to the fact that $s(g_0 + b(x - g_0)) \in P_G(g_0 + b(x - g_0))$. Therefore we have that $s(g_0 + b(x - g_0)) \in S(x, d(x, G))$ and (1) (1) holds.

Since by an observation of Klee [12] (for a proof see Brosowski and Deutsch [6]) the set $S(x, d(x, G)) \cap S(g_0 + b(x - g_0), d(g_0 + b(x - g_0), G))$ is star shaped about g_0, from (1) it follows that for each $0 \leqslant b \leqslant 1$ $ag_0 + (1 - a)\, s(g_0 + b(x - g_0)) \in S(x, d(x, G))$ $(0 \leqslant a \leqslant 1)$.

Therefore for each $0 \leqslant b \leqslant 1$

(4) $\|x - (ag_0 + (1 - a)\, s(g_0 + b(x - g_0))\| = d(x, G)(0 \leqslant a \leqslant 1)$.

Now let (x_n) be a sequence in $\{g_0 + b(x - g_0): 0 \leqslant b \leqslant 1\}$, i.e., $x_n = g_0 + b_n(x - g_0)$ with $0 \leqslant b_n \leqslant 1$, which converges to a point $x \in E$. Then by (4) for each n we have

$$\|x - (ag_0 + (1 - a)\, s(x_n))\| = d(x, G).$$

Since s is inner-radial-continuous and (x_n) converges to x we have

$$\|x - (ag_0 + (1 - a)\, s(x))\| = d(x, G).$$

This is true for each $0 \leqslant a \leqslant 1$ and therefore $ag_0 + (1 - a)\, s(x)$ is in $S(x, d(x, G))$, i.e., $[g_0, s(x)] \subset S(x, d(x, G))$.

This completes the proof.

Theorem 2.1 has been proved for continuous selections in Nürnberger [17].

2.2. COROLLARY. *Let G be the boundary of a ball in a normed linear space E. Then there exists no inner-radial-continuous (in particular no continuous) selection s for P_G.*

Proof. Let $G = S(x_0, r) = \{g \in E: \|x_0 - g\| = r\}$ for some $x_0 \in E$ and $r > 0$. Then G is proximinal, since for each $x \in E$ we have $g_0 \in P_G(x)$, where $g_0 = x_0 + (r/\|x - x_0\|)(x - x_0)$, because $\|x - g_0\| = \|x - x_0 - (r/\|x - x_0\|)(x - x_0)\| = \|x - x_0\| - r = \|x - x_0\| - \|x_0 - g\| \leqslant \|x - g\|$ for each $g \in G$. Since $P_G(x_0) = G$ we have that $s(x_0)$ and $2x_0 - s(x_0)$

are in $P_G(x_0)$ but obviously $[2x_0 - s(x_0), s(x_0)] \not\subset S(x_0, d(x_0, G))$. By Theorem 2.1 we get a contradiction. This completes the proof.

Furthermore using Theorem 2.1 it easily follows that a proximinal subset G in a strictly convex space admits an inner-radial-continuous selection for P_G if and only if G is Chebyshev. This result can be applied to the generalized rational functions $R_{m,n}$ in L_p-spaces ($1 < p < \infty$), which are always proximinal, but Chebyshev if and only if $n = 0$ (see Blatter [1] and Efimov and Stechkin [8]).

Next we consider the metric projection for exponential sums. An *exponential sum* is a function $g \in C[a, b]$ which can be represented as $g(x) = \sum_{i=1}^{l} p_i(x) e^{t_i x}$, where $p_i \in C[a, b]$ is a polynomial of degree d_i and $t_1, ..., t_l$ are distinct. The integer $\sum_{i=1}^{l} (d_i - 1)$ is called the *degree* of g. By E_n we denote the set of all exponential sums with degree less of equal to n.

In contrary to the rational functions and the usual exponential sums, which are Chebyshev in $C[a, b]$ (see Meinardus [15]), the exponential sums E_n, as been defined here, are proximinal but not Chebyshev in $C[a, b]$ for $n \geq 2$. (see Braess [2, pp. 315]). They represent a frequently investigated nonlinear class of functions.

The next result gives a characterization of inner-radial-continuous selections for P_{E_n}.

2.3. THEOREM. *The metric projection from $C[a, b]$ onto the set of exponential sums E_n has an inner-radial-continuous selection if an only if $n = 1$.*

Proof. If $n = 1$ then E_n is Chebyshev and therefore the metric projection P_{E_n} has an inner-radial-continuous selection. If $n \geq 2$ then from the proof of Theorem 8.7 in Braess [3] it can be seen that there exists a continuously differentiable function $f \in C[a, b]$, which has two distinct best approximations g_1 and g_2 in $P_{E_n}(f)$. We construct two sequences (f_m) (respectively, $(\tilde{f}_m)$), which are in $\{g_1 + a(f - g_1) : 0 \leq a \leq 1\}$ (respectively, in $\{g_2 + a(f - g_2) : 0 \leq a \leq 1\}$) such that $f_m \to f$, $\tilde{f}_m \to f$ and $P_{E_n}(f_m) = \{g_1\}$ (respectively, $P_{E_n}(\tilde{f}_m) = \{g_2\}$). This shows that there does not exist an inner-radial-continuous selection for P_{E_n}, because if there were an inner-radial-continuous selection s for P_{E_n}, then we would have $s(f_m) = g_1$ and $s(\tilde{f}_m) = g_2$ for each m and, since $f_m \to f$ and $\tilde{f}_m \to f$, $s(f) = g_1$ and $s(f) = g_2$. But this is impossible, since $g_1 \neq g_2$.

We define for each m functions $f_m := g_1 + (1 - 1/m)(f - g_1)$ and $\tilde{f}_m := g_2 + (1 - 1/m)(f - g_2)$. We show that $P_{E_n}(f_m) = \{g_1\}$ for each m. Since $g_1 \in P_{E_n}(f)$, by the proof of Lemma 2.1 in Singer [21] $g_1 \in P_{E_n}(f_m)$ for each m. Assume there exists a function $\tilde{g}_1 \in P_{E_n}(f_m)$ with $\tilde{g}_1 \neq g_1$. Then $\tilde{g}_1 \in P_{E_n}(f)$ because, if $\|f - g_1\| < \|f - \tilde{g}_1\|$, then $(1 - (1/m))\|f - g_1\| = \|f_m - g_1\| = \|f_m - \tilde{g}_1\| = \|g_1 + (1 - (1/m))(f - g_1) - \tilde{g}_1\| = \|(f - \tilde{g}_1) - (1/m)(f - g_1)\| \geq \|f - \tilde{g}_1\| - (1/m)\|f - g_1\| > \|f - g_1\| - (1/m)\|f - g_1\| = (1 -

$(1/m))\|f - g_1\|$, which is a contradicion. By Satz 1 in Braess [2] there exist $a \leqslant x_0 < \cdots < x_{n+1} \leqslant b$ such that $\epsilon(-1)^i (f - \tilde{g}_1)(x_i) = \|f - \tilde{g}_1\|$, $i = 0,..., n+1$, $\epsilon = \pm 1$. Then $(1 - (1/m))\|f - g_1\| = \|f_m - g_1\| = \|f_m - \tilde{g}_1\| \geqslant \epsilon(-1)^i (f_m - \tilde{g}_1)(x_i) = \epsilon(-1)^i (f - \tilde{g}_1)(x_i) - \epsilon(-1)^i (1/m)(f - g_1)(x_i) = \|f - \tilde{g}_1\| - \epsilon(-1)^i (1/m)(f - g_1)(x_i) = \|f - g_1\| - \epsilon(-1)^i (1/m)(f - g_1)(x_i) \geqslant \|f - g_1\| - (1/m)\|f - g_1\| = (1 - (1/m))\|f - g_1\|$.

Now $\epsilon(-1)^i (f - g_1)(x_i) = \|f - g_1\| = \|f - \tilde{g}_1\| = \epsilon(-1)^i (f - \tilde{g}_1)(x_i)$ and therefore $(g_1 - \tilde{g}_1)(x_i) = 0$, $i = 0,..., n$. Since the points $a \leqslant x_0 < \cdots < x_{n-1} \leqslant b$ are extreme points of $f - g_1$ and $f - \tilde{g}_1$, we have $(f' - g_1')(x_i) = 0 = (f' - \tilde{g}_1')(x_i)$, $i = 1,..., n$. Then $g_1 - \tilde{g}_1$ has at least $2n$ zeros, counting multiplicities, and at most degree $2n$, but by Meinardus [15, pp. 167], this is impossible. Therefore $P_{E_n}(f_m) = \{g_1\}$ and analogously, $P_{E_n}(\tilde{f}_m) = \{g_2\}$ for each m. This completes the proof.

References

1. J. Blatter, Approximative Kompaktheit verallgemeinerter rationaler Funktionen, *J. Approximation Theory* **1** (1968), 85–93.
2. D. Braess, Approximation mit Exponentialsummen, *Computing* **2** (1967), 309–321.
3. D. Braess, Chebyshev approximation with γ-polynomials, *J. Approximation Theory* **9** (1973), 20–43.
4. J. Bram, Chebyshev approximation in locally compact spaces, *Proc. Amer. Math. Soc.* **9** (1958), 133–136.
5. B. Brosowski and F. Deutsch, On some geometric properties of suns, *J. Approximation Theory* **10** (1974), 245–267.
6. B. Brosowski and F. Deutsch, Radial continuity of set-valued metric projections, *J. Approximation Theory* **11** (1974), 236–253.
7. A. L. Brown, On continuous selections for metric projections in spaces of continuous functions, *J. Functional Analysis* **8** (1971), 431–449.
8. F. Deutsch, G. Nürnberger, and I. Singer, *Pacific J. Math.*, to appear.
9. N. V. Efimov and S. B. Stechkin, Approximative compactness and Chebyshev sets, *Dokl. Akad. Nauk SSSR* **140** (1961), 522–524; *Soviet. Math. Dokl.* **2** (1961), 1226–1228.
10. R. B. Holmes, "A Course on Optimization and Best Approximation," Lecture Notes No. 257, Springer–Verlag, Berlin Heidelberg New York, 1972.
11. R. C. Jones and L. A. Karlovitz, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approximation Theory* **3** (1970), 138–145.
12. V. Klee, Remarks on nearest points in normed linear spaces, *in* "Proceedings, Colloquium on Convexity, Copenhagen, 1965, pp. 168–176.
13. A. J. Lazar, P. D. Morris, and D. E. Wulbert, Continuous selections for metric projections, *J. Functional Analysis* **3** (1969), 193–216.
14. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
15. G. Meinardus, Approximation von Funktionen und ihre numerische Behandlung, Springer–Verlag, Berlin, 1967.
16. G. Nürnberger, Nonexistence of continuous selections for the metric projection, preprint.
17. G. Nürnberger, Schnitte für die metrische Projektion, *J. Approximation Theory* **20** (1977), 196–220.

18. G. NÜRNBERGER AND M. SOMMER, Weak Chebyshev subspaces and continuous selections for the metric projection, *Trans. Amer. Math. Soc.*, to appear.
19. G. NÜRNBERGER AND M. SOMMER, Characterization of continuous selections of the metric projection for spline functions, *J. Approximation Theory*, to appear.
20. E. SCHMIDT, On the continuity of the set-valued exponential metric projection, *J. Approximation Theory* **7** (1973), 36–40.
21. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer–Verlag, Berlin, 1970.
22. I. SINGER, "The Theory of Best Approximation and Functional Analysis," Society for Industrial and Applied Mathematics, Philadelphia, 1974.
23. M. SOMMER, Nonexistence of continuous selections of the metric projection for a class of weak Chebyshev spaces, preprint.
24. M. SOMMER AND H. STRAUSS, Eigenschaften von schwach tschebyscheffschen Räumen, *J. Approximation Theory* **21** (1977), 257–268.
25. L. P. VLASOV, Approximative properties of sets in normed linear spaces, *Russian Math. Surveys* **28** (1973), 1–66.
26. R. ZIELKE, Einige Eigenschaften schwacher Tschebyscheff-Systeme, preprint.